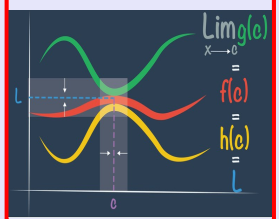
Calculus I

Lecture 24



Feb 19-8:47 AM

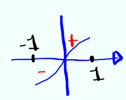
Bonus Class QZ:

Find $\boxed{f_{ave}}$ of the function $f(x)=x(x^2+1)^3$ over the interval $[-1,1]$.
a b

$f(-x) = -x((-x)^2+1)^3$

$= -x(x^2+1)^3$

$= -f(x)$



f(x) is an odd func.

$\int_{-a}^{a} f(x)\,dx = \boxed{0}$

$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$

$= \frac{1}{1-(-1)}\int_{-1}^{1} x(x^2+1)^3\,dx$

$= \frac{1}{2}\int_{-1}^{1} x(x^2+1)^3\,dx = \frac{1}{2}\cdot 0 = \boxed{0}$

$u = x^2+1$ $\quad du = 2x\,dx$

$x=1 \to u=2$ $\quad \frac{du}{2} = x\,dx$

$x=-1 \to u=2$

$= \frac{1}{2}\int_2^2 u^3\,\frac{du}{2}$

$= \frac{1}{4}\int_2^2 u^3\,du = \frac{1}{4}\cdot 0 = \boxed{0}$

$\int_a^a f(x)\,dx = 0$

$= \frac{1}{4}\,\frac{u^4}{4}\Big|_2^2 = \frac{1}{16}\left[2^4-2^4\right] = \frac{1}{16}(16-16) = \boxed{0}$

Jul 28-7:22 AM

Find f_{ave} for $f(x)=\dfrac{x}{(x^2+1)^3}$ on $[0,2]$.

$$f_{ave}=\frac{1}{b-a}\int_a^b f(x)\,dx \qquad u=x^2+1 \to \frac{du}{2}=x\,dx$$

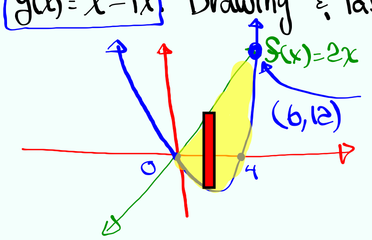
$$=\frac{1}{2-0}\int_0^2 \frac{x}{(x^2+1)^3}\,dx=\frac{1}{2}\int_1^5 \frac{1}{u^3}\,\frac{du}{2} \qquad \begin{array}{ll} x=0 & u=1 \\ x=2 & u=5 \end{array}$$

$$=\frac{1}{4}\int_1^5 u^{-3}\,du=\frac{1}{4}\cdot\frac{u^{-3+1}}{-3+1}\Big|_1^5=\frac{-1}{8}\,\frac{1}{u^2}\Big|_1^5$$

$$=\frac{-1}{8}\left[\frac{1}{5^2}-\frac{1}{1^2}\right]=\frac{-1}{8}\left[\frac{1}{25}-1\right]=\frac{-1}{\cancel{8}}\cdot\frac{-\cancel{24}^{\,3}}{25}=\boxed{\frac{3}{25}}$$

Jul 29-8:22 AM

Find the area bounded by $\boxed{f(x)=2x}$ and $\boxed{g(x)=x^2-4x}$. Drawing & labeling Required.



$$g(x)=f(x)$$
$$x^2-4x=2x$$
$$x^2-4x-2x=0$$
$$x^2-6x=0$$
$$x(x-6)=0$$
$$x=0 \qquad x=6$$

$$A=\int_0^6\Big[\overset{\substack{2x \\ \text{Top}}}{2x}\;\overset{\substack{-\,(x^2-4x) \\ \text{Bottom}}}{-x^2+4x}\Big]\,dx$$

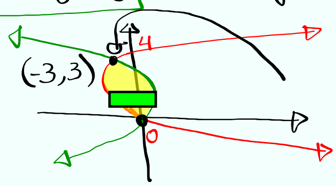
$$=\int_0^6\left[6x-x^2\right]dx=\left(\frac{6x^2}{2}-\frac{x^3}{3}\right)\Big|_0^6$$

$$=\left(3x^2-\frac{x^3}{3}\right)\Big|_0^6$$

$$=\left(3\cdot 6^2-\frac{6^3}{3}\right)-\left(3\cdot 0^2-\frac{0^3}{3}\right)=\boxed{36}$$

Jul 29-8:29 AM

Find the area bounded by $\boxed{x = y^2 - 4y}$ and $\boxed{x = 2y - y^2}$. Drawing & Labeling Required.



$$\boxed{y^2 - 4y} = \boxed{2y - y^2}$$

$$y^2 - 4y - 2y + y^2 = 0$$

$$2y^2 - 6y = 0$$

$$y^2 - 3y = 0$$

$$y(y-3) = 0$$

$$y = 0 \quad y = 3$$

$$A = \int_0^3 \overset{\text{Right - left}}{(2y - y^2 - y^2 + 4y)}\, dy$$

$$= \int_0^3 (6y - 2y^2)\, dy$$

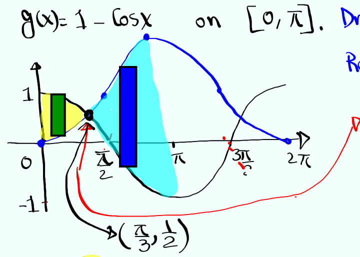
$$= \left(\frac{6y^2}{2} - \frac{2y^3}{3}\right)\Big|_0^3$$

$$= \left(3y^2 - \frac{2}{3}y^3\right)\Big|_0^3 = 3 \cdot 3^2 - \frac{2}{3} \cdot 3^3 - 0$$

$$= 27 - 18 = \boxed{9}$$

Jul 29-8:38 AM

Find the area bounded by $f(x) = \cos x$ and $g(x) = 1 - \cos x$ on $[0, \pi]$. Drawing & Labeling Required.



$$\cos x = 1 - \cos x$$

$$2\cos x = 1$$

$$\cos x = \frac{1}{2}$$

$$x =$$

$$A = \int_0^{\pi/3} [\cos x - 1 + \cos x]\, dx + \int_{\pi/3}^{\pi} [1 - \cos x - \cos x]\, dx$$

Top $= \cos x$ Bottom $= 1 - \cos x$

Top $= 1 - \cos x$ Bottom $= \cos x$

$$= \int_0^{\pi/3} (2\cos x - 1)\, dx + \int_{\pi/3}^{\pi} [1 - 2\cos x]\, dx$$

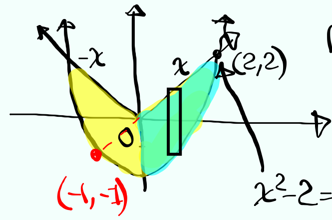
$$= (2\sin x - x)\Big|_0^{\pi/3} + (x - 2\sin x)\Big|_{\pi/3}^{\pi}$$

$$= \left(2\sin\frac{\pi}{3} - \frac{\pi}{3}\right) - \left(2\cancel{\sin 0}^{\,0} - 0\right) + \left(\pi - 2\cancel{\sin\pi}^{\,0}\right) - \left(\frac{\pi}{3} - 2\sin\frac{\pi}{3}\right)$$

$$= \cancel{2 \cdot \frac{\sqrt{3}}{2}} - \frac{\pi}{3} + \pi - \frac{\pi}{3} - \cancel{2 \cdot \frac{\sqrt{3}}{2}} = \pi - \frac{2\pi}{3} = \boxed{\frac{\pi}{3}}$$

Jul 29-8:47 AM

Find the area bounded by $f(x)=|x|$ and $g(x)=x^2-2$. Drawing, Labeling Required.



Top − Bottom

$$A=2\int_0^2 (x-x^2+2)\,dx$$

$$x^2-2=x$$
$$x^2-x-2=0$$
$$(x-2)(x+1)=0$$
$$x=2 \quad x=-1$$

$$=2\left[\frac{x^2}{2}-\frac{x^3}{3}+2x\right]\Bigg|_0^2$$

$$=2\left[\frac{2^2}{2}-\frac{2^3}{3}+2(2)\right]-0$$

$$=2\left[2-\frac{8}{3}+4\right]$$

$$=2\left[6-\frac{8}{3}\right]=2\cdot\frac{10}{3}$$

$$=\boxed{\frac{20}{3}}$$

Jul 29-9:04 AM

Use Reimann Sum to evaluate $\int_0^2 (3x^2-2x)\,dx$

$$\int_a^b f(x)\,dx=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\cdot\Delta x \qquad =(x^3-x^2)\Big|_0^2$$

$a=0$, $b=2$ $\quad f(x)=3x^2-2x \qquad =2^3-2^2=\boxed{4}$

$$\Delta x=\frac{b-a}{n}=\frac{2-0}{n}=\frac{2}{n} \qquad x_i=a+i\Delta x=0+i\cdot\frac{2}{n}=\frac{2i}{n}$$

$$f(x_i)=3(x_i)^2-2(x_i)=3\cdot\left(\frac{2i}{n}\right)^2-2\left(\frac{2i}{n}\right)$$
$$=\frac{12i^2}{n^2}-\frac{4i}{n}$$

$$f(x_i)\cdot\Delta x=\left(\frac{12i^2}{n^2}-\frac{4i}{n}\right)\cdot\frac{2}{n}=\frac{24i^2}{n^3}-\frac{8i}{n^2}$$

$$\sum_{i=1}^{n} f(x_i)\Delta x=\sum_{i=1}^{n}\left[\frac{24i^2}{n^3}-\frac{8i}{n^2}\right]$$

$$=\sum_{i=1}^{n}\frac{24i^2}{n^3}-\sum_{i=1}^{n}\frac{8i}{n^2}$$

$$=\frac{24}{n^3}\boxed{\sum_{i=1}^{n} i^2}-\frac{8}{n^2}\boxed{\sum_{i=1}^{n} i}$$

$$=\frac{24}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}-\frac{8}{n^2}\cdot\frac{n(n+1)}{2}$$

$$=\frac{48n^3+\cdots\cdots}{6n^3}-\frac{8n^2+\cdots}{2n^2}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x=\lim_{n\to\infty}\left[\frac{48n^3+\cdots}{6n^3}-\frac{8n^2+\cdots}{2n^2}\right]$$

$$=\frac{48}{6}-\frac{8}{2}=8-4=\boxed{4}$$

Jul 29-9:15 AM

Evaluate $\int_0^1 (x+1)\sqrt{x^2+2x}\,dx$

$x=0 \Rightarrow u=0$

$x=1 \Rightarrow u=3$

Use Subs. Method

$u = x^2+2x$

$du = (2x+2)\,dx$

$du = 2(x+1)\,dx$

$\frac{du}{2} = (x+1)\,dx$

$$= \int_0^3 \sqrt{u}\,\frac{du}{2} \qquad u^{1/2}$$

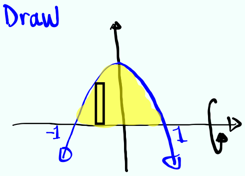
$$= \frac{1}{2}\cdot\frac{u^{3/2}}{3/2}\Bigg|_0^3$$

$$= \frac{1}{3}\,u\sqrt{u}\,\Bigg|_0^3 = \frac{1}{3}\cdot 3\sqrt{3} = \boxed{\sqrt{3}}$$

Jul 29-9:28 AM

Consider the region bounded by $y=1-x^2$ and $y=0$. x-axis

1) Draw



2) find its area.

$$A=\int_{-1}^{1}(1-x^2-0)\,dx = 2\int_0^1 (1-x^2)\,dx$$

$$= 2\left[x-\frac{x^3}{3}\right]\Bigg|_0^1 = \boxed{\frac{4}{3}}$$

3) Rotate this region by x-axis, find the Volume.

a) Region attached totally to A.O.R.

b) Ref. Rect. $\perp$ A.O.R.

$\Rightarrow$ **Disk**

$R = \text{Top} - \text{Bottom}$
$= 1-x^2-0$
$= 1-x^2$

$$V=\int_{-1}^{1}\pi(1-x^2)^2\,dx = 2\int_0^1 \pi(1-2x^2+x^4)\,dx$$

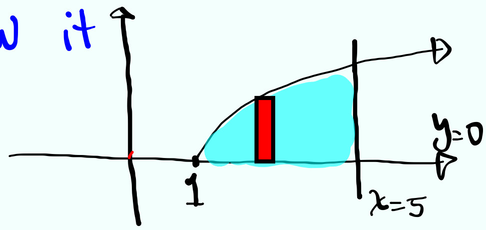
$$= 2\pi\left[x-\frac{2x^3}{3}+\frac{x^5}{5}\right]\Bigg|_0^1 = 2\pi\left[1-\frac{2}{3}+\frac{1}{5}\right] = \boxed{\frac{16\pi}{15}}$$

Jul 29-9:56 AM

Consider the region bounded by

$y = \sqrt{x-1}$, $y = 0$, and $x = 5$.

1) Draw it



2) Find its area.

$$A = \int_1^5 \left[\sqrt{x-1} - 0\right] dx$$

$$= \int_1^5 \sqrt{x-1}\, dx$$

$u = x-1$

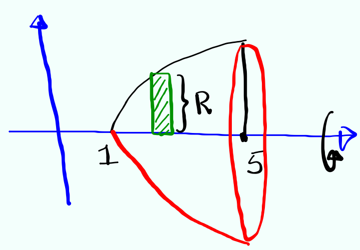
$du = dx$

$\sqrt{u} \to u^{1/2}$

$$= \int_0^4 \sqrt{u}\, du = \frac{u^{3/2}}{3/2}\Big|_0^4 = \frac{2}{3} u\sqrt{u}\Big|_0^4 = \frac{2}{3}\cdot 4\sqrt{4} = \boxed{\frac{16}{3}}$$

Jul 29-10:05 AM

3) Rotate the region by x-axis , find the Volume



1) Region is totally attached to A.O.R.

2) Ref. Rect. $\perp$ A.O.R.

Disk

$$V = \int_1^5 \pi\left[\sqrt{x-1}\right]^2 dx = \pi\int_1^5 (x-1)dx = \pi\left[\frac{x^2}{2} - x\right]\Big|_1^5$$

$$= \pi\left[\left(\frac{25}{2} - 5\right) - \left(\frac{1}{2} - 1\right)\right]$$

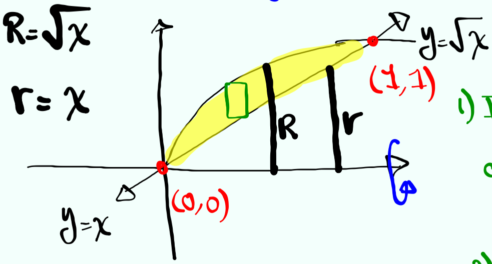
$$= \pi\left[\frac{24}{2} - 4\right] = \pi\cdot 8 = \boxed{8\pi}$$

Jul 29-10:15 AM

Rotate the region below by x-axis. Find its Volume.

$R = \sqrt{x}$

$r = x$



1) Is the region totally attached to A.O.R.?

NO

2) Is Ref. Rect. ⊥ A.O.R

Yes

washer Method

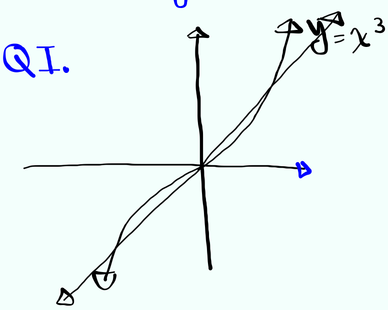
$$V = \int_a^b \pi \left[R^2 - r^2 \right] dx = \int_0^1 \pi \left[(\sqrt{x})^2 - x^2 \right] dx$$

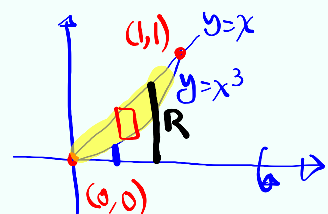
$$= \pi \int_0^1 (x - x^2)\, dx = \pi \left[\frac{x^2}{2} - \frac{x^3}{3} \right] \Big|_0^1$$

$$= \pi \cdot \frac{1}{6} = \boxed{\frac{\pi}{6}}$$

Jul 29-10:21 AM

Draw the region bounded by $y = x^3$ and $y = x$ in QI.





$$A = \int_0^1 \left[x - x^3 \right] dx$$

$$= \left(\frac{x^2}{2} - \frac{x^4}{4} \right) \Big|_0^1 = \boxed{\frac{1}{4}}$$

Rotate by x-axis, find Volume.

washer $\quad R = x \quad r = x^3$

$$V = \int_0^1 \pi \left[x^2 - (x^3)^2 \right] dx = \pi \left(\frac{x^3}{3} - \frac{x^7}{7} \right) \Big|_0^1$$

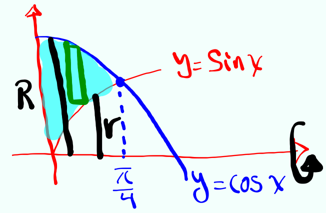
$$= \boxed{\frac{4\pi}{21}}$$

Jul 29-10:28 AM

Rotate the region bounded by $x=0$ (y-axis), $x=\frac{\pi}{4}$, $y=\sin x$ and $y=\cos x$ about x-axis.

Find the Volume.

washer



$R=\cos x$ $\quad r=\sin x$

$$V=\int_0^{\pi/4}\pi\left[R^2-r^2\right]dx$$

$$=\pi\int_0^{\pi/4}\underbrace{(\cos^2 x-\sin^2 x)}_{\cos 2x}dx=\pi\int_0^{\frac{\pi}{4}}\cos 2x\,dx$$

$u=2x$

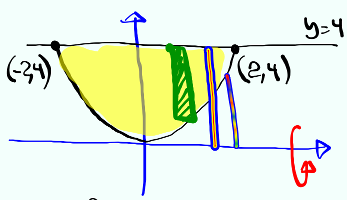
$du=2dx$

$\frac{du}{2}=dx$

$$=\pi\int_0^{\pi/2}\cos u\,\frac{du}{2}=\frac{\pi}{2}\sin u\Big|_0^{\pi/2}$$

$$=\frac{\pi}{2}\left[\cancelto{1}{\sin\frac{\pi}{2}}-\cancelto{0}{\sin 0}\right]=\boxed{\frac{\pi}{2}}$$

Rotate the region bounded by $y=x^2$ and $y=4$ about x-axis. Find its Volume.



washer

$R=4$

$r=x^2$

$$V=\int_{-2}^{2}\pi\left[4^2-(x^2)^2\right]dx=\pi\cdot 2\int_0^2\left[16-x^4\right]dx$$

$$=2\pi\left[16x-\frac{x^5}{5}\right]\Big|_0^2=2\pi\left[16\cdot 2-\frac{2^5}{5}-0\right]$$

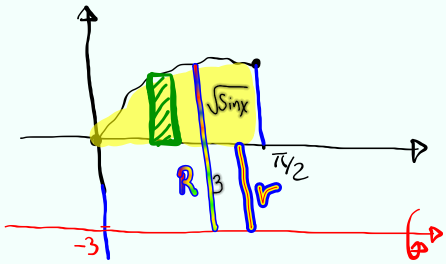
$$=2\pi\left[32-\frac{32}{5}\right]=2\pi\cdot\frac{128}{5}$$

$$=\boxed{\frac{256\pi}{5}}$$

Rotate the region bounded by $y=\sqrt{\sin x}$, $y=0$ for $0 \le x \le \frac{\pi}{2}$ about $y=-3$.

Find its volume.

washer



$R = 3+\sqrt{\sin x}$ $\quad r=3$

$$\boxed{V = \int_0^{\pi/2} \pi\left[(3+\sqrt{\sin x})^2 - 3^2\right]dx}$$

Set-up the integral

$$= \pi\int_0^{\pi/2}\left[\cancel{9} + 6\sqrt{\sin x} + \sin x - \cancel{9}\right]dx$$

More tools

Jul 29-10:54 AM